A Petrov-Galerkin method for flows in cavities

Sergey A. Suslov[a] **and Samuel Paolucci**[b]
[a]*Department of Mathematics and Computing*
University of Southern Queensland, Toowoomba, Queensland 4350, Australia
[b]*Department of Aerospace and Mechanical Engineering*
University of Notre Dame, Notre Dame, Indiana 46556, USA

Keywords: Convection, spectral methods, Petrov-Galerkin method, differentially-heated cavity

1. Abstract

A Petrov-Galerkin method for computations of fully enclosed flows is developed. It makes use of divergence-free basis functions which also satisfy the boundary conditions for the velocity field. This allows the elimination of the unknown pressure. The computational procedure reduces to the solution of a system of nonlinear first order ordinary differential equations for the spectral expansion coefficients. We illustrate the effectiveness of the method by solving the problem of the two-dimensional thermoconvective flow in a rectangular cavity of aspect ratio 8 at Rayleigh number 3.4×10^5.

2. Introduction

Spectral and, in particular, Galerkin-type methods are usually the first method of choice when high accuracy of results is required in the solution of partial differential equations. But when a very large number of spectral modes is necessary to resolve the fine structures of the solution, the application of spectral methods becomes extremely time consuming.

The proposed method (see [1]) takes into account some of the properties of the flow and incorporates them into the construction of specific basis functions. This guarantees a faster convergence rate than obtained using conventional spectral methods.

3. Problem formulation

We consider the two dimensional flow of a fluid in a rectangular enclosure. The dimensionless governing equations, corresponding to conservations of mass, momentum and energy, under the Boussinesq approximation, are given by

$$\nabla \cdot \vec{u} = 0, \tag{1}$$
$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla P - \theta \vec{n} + \sqrt{\frac{Pr}{Ra}}\,\nabla^2 \vec{u}, \tag{2}$$
$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = \frac{1}{\sqrt{PrRa}}\,\nabla^2 \theta, \tag{3}$$

where $\vec{u} = (u, v, 0)^T$ is the velocity vector, $P = p + |\vec{u}|^2/2$ is the total pressure, and $\vec{n} = (0, -1, 0)^T$ is the unit vector in the direction of gravity. Note that it is advantageous to introduce the vorticity vector $\vec{\omega} = (0, 0, \partial v/\partial x - \partial u/\partial y)^T$ since in this case only two velocity derivatives enter the convective terms of the momentum equations instead of four.

Boundary conditions at the walls are given by

$$\begin{gathered} \vec{u} = 0 \quad \text{at } x = 0, 1 \text{ and } y = 0, A, \\ \theta = \pm 1/2 \ \text{ at } x = 0, 1 \text{ and } \frac{\partial \theta}{\partial y} = 0 \text{ at } y = 0, A. \end{gathered} \tag{4}$$

The initial conditions used are $\vec{u} = 0$ and $\theta = 1/2 - x$.

The independent dimensionless parameters appearing in the problem are respectively the Rayleigh number, the Prandtl number, and the aspect ratio:

$$Ra = \frac{\beta g \triangle T W^3}{\nu \alpha}, \; Pr = \frac{\nu}{\alpha}, \; A = \frac{H}{W}. \tag{5}$$

4. Method of solution

To solve the system of Boussinesq equations (1–3) we require our expansion bases to satisfy the following criteria: 1) the basis functions for velocity should be divergence-free, 2) the bases functions should satisfy homogeneous boundary conditions, 3) the bases functions should be complete, and 4) the resulting system of equations should have sparse matrices. As a result of these criteria, and because of the simple geometry, we are led to the following expansions for the unknown functions:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{i=0}^{K}\sum_{j=0}^{L} C_{ij}(t) \begin{pmatrix} g_i(x)\cdot h'_j\left(\frac{y}{A}\right) \\ -Ag'_i(x)\cdot h_j\left(\frac{y}{A}\right) \end{pmatrix},$$
$$\theta = \frac{1}{2} - x + \sum_{i=0}^{M}\sum_{j=0}^{N} D_{ij}(t)\cdot p_i(x)\cdot r_j\left(\frac{y}{A}\right), \tag{6}$$

where, upon using the transformations $\tilde{x} = 2x-1, \;\; \tilde{y} = 2y/A - 1$, we have for $i, j = 0, 1, 2, \ldots$

$$g_i(T(\tilde{x})) = T_i(\tilde{x}) - 2\left(\frac{i+2}{i+3}\right)T_{i+2}(\tilde{x}) + \left(\frac{i+1}{i+3}\right)T_{i+4}(\tilde{x}),$$
$$h_j(T(\tilde{y})) = T_j(\tilde{y}) - 2\left(\frac{j+2}{j+3}\right)T_{j+2}(\tilde{y}) + \left(\frac{j+1}{j+3}\right)T_{j+4}(\tilde{y}),$$
$$p_i(T(\tilde{x})) = T_i(\tilde{x}) - T_{i+2}(\tilde{x}),$$
$$r_j(T(\tilde{y})) = T_j(\tilde{y}) - \left(\frac{j}{j+2}\right)^2 T_{j+2}(\tilde{y}). \tag{7}$$

Note that $T_i(\tilde{x}) = \cos\left(i \;\arccos\left(\tilde{x}\right)\right)$ is the Chebyshev polynomial of the first type.

After the choice of bases, we use a Petrov-Galerkin procedure with the following test functions:

$$\vec{f}_{mn} = \begin{pmatrix} g_m\left(T(\tilde{x})/\sqrt{1-\tilde{x}^2}\right)\cdot h'_n\left(T(\tilde{y})/\sqrt{1-\tilde{y}^2}\right) \\ -Ag'_m\left(T(\tilde{x})/\sqrt{1-\tilde{x}^2}\right)\cdot h_n\left(T(\tilde{y})/\sqrt{1-\tilde{y}^2}\right) \end{pmatrix},$$
$$q_{mn} = p_m\left(T(\tilde{x})/\sqrt{1-\tilde{x}^2}\right)\cdot r_n\left(T(\tilde{y})/\sqrt{1-\tilde{y}^2}\right). \tag{8}$$

We note that these functions are chosen to satisfy the boundary conditions and in addition that the velocity test functions be divergence-free. Subsequently, we take the inner product of the momentum equation with the test function $\vec{f}_{mn}$ and integrate over the volume. We also integrate the thermal energy equation over the volume after multiplying it by the test function q_{mn}. We note that for an arbitrary scalar function P and a divergence-free vector field $\vec{f}_{mn}$ which satisfies zero boundary conditions,

$$\int\limits_V \nabla P\cdot\vec{f}_{mn}\,dV = -\int\limits_V P\nabla\cdot\vec{f}_{mn}dV + \int\limits_S P\vec{f}_{mn}\cdot d\vec{S} \equiv 0, \tag{9}$$

where V and S denote the volume and surface of the domain, respectively. Thus, after application of the Petrov-Galerkin procedure, the unknown total pressure is eliminated, and we obtain a system of *nonlinear ordinary* differential equations for the spectral amplitudes:

$$\mathbf{S}\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{F}, \tag{10}$$

where

$$\mathbf{X} = \begin{bmatrix} C \\ D \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} S_{ff} & 0 \\ 0 & S_{qq} \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 4\sqrt{Pr/Ra}A_{ff} & A_{fq} \\ A_{qf} & 1/\sqrt{Ra\,Pr}A_{qq} \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} C^T B_{fff} C + \ F_{ff} \\ C^T B_{qfq} D \end{bmatrix}. \tag{11}$$

Subscripts denote products of the basis functions participating in the Petrov-Galerkin integration. The coefficients A_{ff} , A_{fq} , A_{qf} , A_{qq} , B_{fff} , B_{qfq} , and F_{ff} are the inner products of Chebyshev polynomials which can be obtained analytically using integration by parts and the orthogonality property of Chebyshev polynomials.

By re-ordering the equations in the system, it is possible to rearrange the sparse matrices $\mathbf{S}$ and $\mathbf{A}$ to block cyclic form. The problem can then be split into two parts coupled through the nonlinear terms in the vector $\mathbf{F}$, the structure of which we have not shown explicitly. Only one component of the submatrix F_{ff} corresponding to $i = j = 0$ is nonzero due to the special choice of the test functions and the orthogonality property of Chebyshev polynomials. Through its effect on the first linearly decoupled sub-block, the top half of this vector contributes in the establishment of the centro-symmetric solution. When Ra is small enough (less than a critical value), the bottom half of the nonlinear vector is identically zero! Subsequently, the second sub-block, which is linearly decoupled, effectively becomes fully decoupled and just has the trivial solution

$$C_{ij} = 0 \ \text{ if } i + j \ \text{ is } odd \quad \text{and} \quad D_{ij} = 0 \ \text{ if } i + j \ \text{ is } even. \tag{12}$$

At higher values of Ra, due to instability to asymmetric disturbances arising from machine round-off, the bottom half of this vector becomes nonzero and the nonlinear coupling between the two sub-blocks becomes so strong that the symmetry of the solution is broken. It is noted that the splitting of the original problem makes the linear stability analysis of the solutions very natural and straightforward, although it will not be discussed here.

The special form of matrices $\mathbf{A}$ and $\mathbf{F}$ allows one to solve the problem by partitioning which reduces substantially both the memory consumption and the computational time. Further optimization of the solution procedure is possible if use is made of special sparse matrix solvers for multidiagonal systems. At this point we have not focused our attention on this issue, and subsequently have used a general sparse matrix package (YSMP).

The procedure just described is ideal if one is interested in the low order dynamical system approximating the given problem. In this case all integrals, including triple products entering the nonlinear terms, can be computed analytically and stored. Then the system of first order ordinary differential equations with known constant coefficients can be solved using any appropriate initial value integration technique. When accurate solutions are required for higher values of Rayleigh numbers, the total number of spectral modes becomes sufficiently large. The storage space required for triple product integrals in the two-dimensional case is proportional to N^6, where N is the number of modes in one direction (assumed equal in each direction). Thus memory limitations become extremely restrictive for the proposed method. Alternatively, a direct calculation of triple products at each time step makes the method extremely time consuming. Fortunately, this last difficulty can be resolved since the Chebyshev polynomial basis enables us to use fast Fourier transforms (FFTs) which require only $\mathrm{O}(N^n lnN)$ operations to evaluate nonlinear terms, where n is the spacial dimension of the problem. Algorithmically, we first find the expansions for $\vec{\omega}$, $\vec{u}$, $\nabla\theta$ in terms of Chebyshev polynomials using recurrence differentiation formulae applied to the original expansion. Then we apply the inverse 2D FFT to find the values of $\vec{\omega}$, $\vec{u}$, $\nabla\theta$ at $2N \times 2N$ Gauss-Lobatto collocation points. The choice of $2N$ collocation points in each direction allows us to retain full spectral accuracy since it leads to the exact triple-product integrals. Lastly, we find values of nonlinear terms in physical space using $\mathrm{O}\big(N^2\big)$ operations and subsequently apply the forward 2D FFT to find the expansion of nonlinear terms as Chebyshev series. Since the expansion of the trial functions in terms of Chebyshev polynomials is known, the integral evaluation reduces to the trivial use of orthogonality formulas.

To avoid the solution of the nonlinear equations at each time step, and preserve the stability characteristics of implicit schemes as much as possible, we implement a semi-implicit time integration procedure combining the second order (implicit) Gear method for the linear part and the second order (explicit) Adams-Bashforth method for the nonlinear term. In the application to our matrix equation this becomes

$$\mathbf{X}^{n+1} = (\mathbf{3S} - \mathbf{2}\triangle t\mathbf{A})^{-1}\big[\mathbf{S}\big(4\mathbf{X}^{n} - \mathbf{X}^{n-1}\big) + \mathbf{2}\triangle t\big(\mathbf{2F}^{n} - \mathbf{F}^{n-1}\big)\big], \tag{13}$$

where $\mathbf{I}$ is the identity matrix. Note that if the time step $\triangle t$ is fixed, then the matrix inversion is only done once and the integration procedure reduces to matrix-vector multiplications which can be accomplished efficiently if one takes advantage of the sparseness of matrices $\mathbf{S}$ and $\mathbf{A}$.

5. Results

We apply the Petrov-Galerkin method to the case $Ra = 3.4 \times 10^5$, $Pr = 0.71$, and $A = 8$. We take $K = M = 50$ and $L = N = 50$. The computations are performed on a Sun UltraSPARC 30, Model 300 with a 296 MHz UltraSPARC-II CPU having 128 MBytes of total memory. For the above parameters, the algorithm requires approximately 85 MBytes of memory and the computational cost is approximately 3×10^{-4} sec/time-step/mode. A summary of the computed results are given in Tables 1–3 and Figures 1 and 2. A major observation from the results is that the flow is periodic. Below we report the average, *rms* and dominant frequency observed from the stationary results. We note that while we report the results of a single computation, we have checked the correctness and accuracy of the code by performing a number of other computations with different number of modes, different aspect ratios, and different Rayleigh numbers. For example, a steady solution exists for relatively small values of the Rayleigh number and $A = 1$. The results compare favorably with those obtained by Le Quéré (1991)[2] using a pseudo-spectral Chebyshev algorithm (to the authors' knowledge these are the most accurate published results). With our Petrov-Galerkin algorithm we obtain the same order of accuracy using a substantially smaller total number of modes.

6. Conclusions

The Petrov-Galerkin method used permits the efficient and spectrally accurate solution of incompressible fully enclosed flows using primitive variables. Pressure is eliminated identically from the system of equations by the special choice of the divergence-free basis satisfying homogeneous boundary conditions. Further reduction of the total number of unknown functions is obtained since the two velocity components are represented by one set of spectral coefficients. The even-odd decomposition of the modes is straightforward and allows to partition the problem leading to substantial computational savings. The numerical procedure is made very efficient by computing analytically and storing all necessary inner products and using FFTs for the explicit evaluation of nonlinear terms. Although not shown here, this technique is easily generalized to three dimensions resulting in even more substantial computer storage and computational time savings in comparison with standard computational techniques.

References

[1] S. A. Suslov and S. Paolucci. A Petrov-Galerkin method for the direct simulation of fully enclosed flows. In *Proc. of the ASME Heat Transfer Division, Vol. 4*, HTD–335, pages 39–46, 1996.

[2] P. Le Quéré. Accurate solutions to the square thermally driven cavity at high Rayleigh number. *Comput. Fluids*, 20:29–41, 1991.

FIGURE 1. Stationary results at Point 1 ($x = 0.1810$, $y = 7.3700$).

Quantity	Grid Resolution: 50×50 Time Duration: $500 \leq t \leq 737$ Number of Samples: 965		
	Average	*rms*	Dominant frequency
u	0.05628	0.01944	0.29559
v	0.46181	0.02709	0.29559
θ	0.26552	0.01507	0.29559
ϵ_{12}	0.00465	0.00358	0.29559
ψ	-0.07370	0.00244	0.29559
ω	-2.37139	0.36170	0.29559

FIGURE 2. Stationary wall Nusselt numbers.

Quantity	Grid Resolution: 50×50 Time Duration: $500 \leq t \leq 737$ Number of Samples: 965		
	Average	*rms*	Dominant frequency
Nu_0	4.57930	0.00253	0.29559
Nu_W	-4.57930	0.00253	0.29559

FIGURE 3. Stationary square roots of global kinetic energy, $\hat{u}$, and enstrophy, $\hat{\omega}$.

Quantity	Grid Resolution: 50×50 Time Duration: $500 \leq t \leq 737$ Number of Samples: 965		
	Average	*rms*	Dominant frequency
$\hat{u}$	0.23951	0.00002	0.29559
$\hat{\omega}$	1.50857	0.00056	0.29559

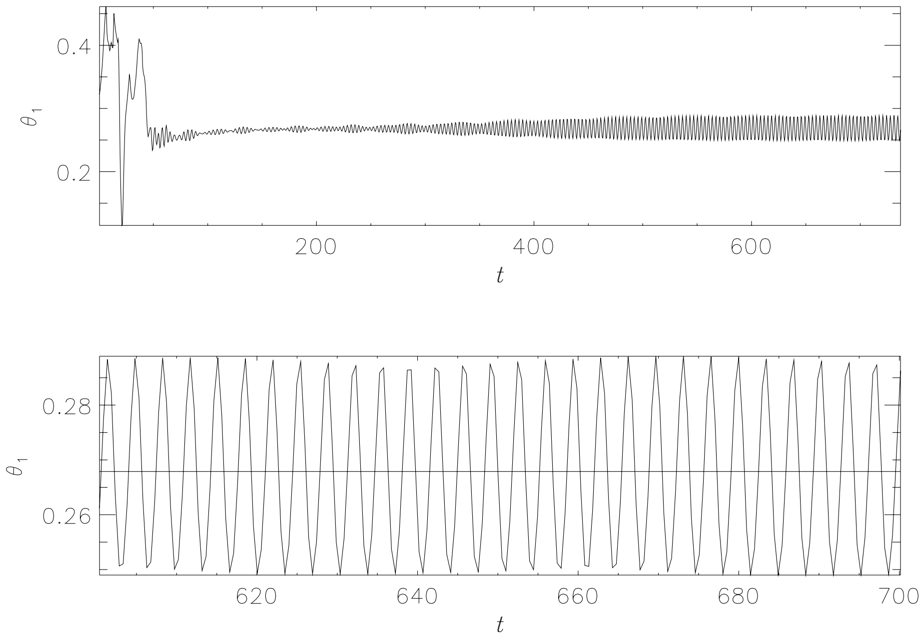

FIGURE 1. History of temperature at Point 1 ($x = 0.1810$, $y = 7.3700$), θ_1, with detail.

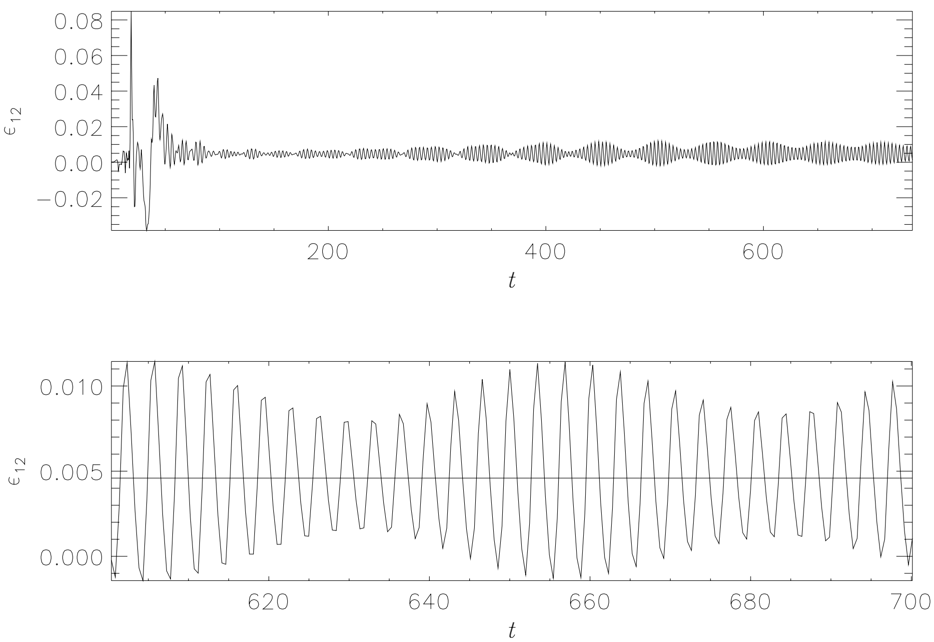

FIGURE 2. History of temperature skewness between Point 1 ($x = 0.1810$, $y = 7.3700$) and Point 2 ($x = 0.8190$, $y = 0.6300$), ϵ_{12}, with detail.